Representation Theory:

A Friendly Introduction

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Preliminaries

What is representation theory?

- Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces
- When such abstract algebraic object is being represented on a finite-dimensional vector space, its elements are described by matrices and its algebraic operations are described by matrix addition and/or matrix multiplication
- Representation theory reduces abstract algebra problems to linear algebra problems

Where is representation theory applied?

- Algebra and number theory
- Category theory
- Quantum physics: the theory of elementary particles and more
- Fourier analysis
- And much more!

Definition

A *group* $(G, \star)$ is a set G equipped with some binary operation $\star\colon G \times G \longrightarrow G, (a, b) \mapsto a \star b$ that satisfies 3 conditions:

- Associativity: $(a \star b) \star c = a \star (b \star c) \quad \forall a, b, c \in G$
- Unitarity: $\exists e \in G$ such that $e \star a = a = a \star e \quad \forall a \in G$ (often we denote $e = 1 = 1_G$)
- Invertibility: $\forall a \in G\ \exists b \in G$ such that $a \star b = e$ and $b \star a = e$ (often we denote $b = a^{-1}$)

Examples

- $(\mathbb{Z}, +)$, $(\Bbbk, +)$, $(\Bbbk^* = \Bbbk \setminus \{0\}, \cdot)$, where $\Bbbk = \mathbb{Q}, \mathbb{R}, \mathbb{C}$
- $(\mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}, +)$
- $(\mathrm{GL}_n(\Bbbk) = \{A \in M_n(\Bbbk) \mid A$ is invertible$\}, \cdot)$
- $(\mathrm{SO}_3(\mathbb{R}) = \{A \in \mathrm{GL}_3(\mathbb{R}) \mid AA^T = I_3 \ \det A = 1\}, \cdot)$
- For any set X, $(S_X = \{\varphi \colon X \to X \mid \varphi$ is bijective$\}, \circ)$; when $X = \{1, 2, \ldots, n\}$, we write $S_X = S_n$
- For any $\Bbbk$-vector space V, $(\mathrm{GL}_{\Bbbk}(V) = \{\varphi \colon V \to V \mid \varphi$ is $\Bbbk$-linear and invertible$\}, \circ)$

Definition

If $(G, \star)$ and $(H, \spadesuit)$ are groups, then a ***group morphism*** $\rho\colon (G, \star) \to (H, \spadesuit)$ is a map $\rho\colon G \to H$ such that $\rho(a \star b) = \rho(a) \spadesuit \rho(b)\ \forall a, b \in G$.

From the group axioms, one can deduce that $\rho(1_G) = \rho(1_H)$ and $\rho(a^{-1}) = \rho(a)^{-1}\ \forall a \in G$.

Examples

- $\iota\colon (\mathbb{R}, +) \to (\mathbb{C}, +), a \mapsto a$
- $\pi\colon (\mathbb{Z}, +) \to (\mathbb{Z}/n\mathbb{Z}, +), a \mapsto \overline{a}$
- $\varphi\colon (G, \star) \to (S_G, \circ), g \mapsto \varphi_g$, where $\varphi_g : a \mapsto g \star a$

Definition

Let $(G, \star)$ be a group and X be a set. A ***group action of*** $(G, \star)$ ***on*** X is a group morphism $\alpha\colon (G, \star) \to (S_X, \circ)$.

So what does this mean:

- $\alpha(1_G) = \mathrm{id}_X$, so $\alpha(1_G)(x) = \mathrm{id}_X(x) = x$ for $x \in X$
- $\alpha(g \star h) = \alpha(g) \circ \alpha(h)$, so $\alpha(g \star h)(x) = \alpha(g)(\alpha(h)(x))$ for $g, h \in G$ and $x \in X$

If we instead use $g \bullet x = \alpha(g)(x)$, then the above conditions may be more familiar:

- $1_G \bullet x = x$ for $x \in X$
- $(g \star h) \bullet x = g \bullet (h \bullet x)$ for $g, h \in G$ and $x \in X$

Example

The group $(D_3, \cdot)$, where $D_3 = \{1, a, a^2, b, ab, a^2b \mid a^3 = b^2 = (ab)^2 = 1\}$, acts on the Triangle by means of symmetry.

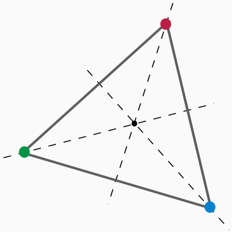

Figure 1: Symmetries of the Triangle

Example

The group $(\mathrm{SO}_3(\mathbb{R}), \cdot)$ acts on the vector space $\mathbb{R}^3$ via matrix multiplication:

$$Ax \in \mathbb{R}^3 \quad \text{for} \quad A \in \mathrm{SO}_3(\mathbb{R}),\ x \in \mathbb{R}^3$$

The group $(\mathrm{SO}_3(\mathbb{R}), \cdot)$ is known as 'the 3D rotation group' because it is the group of all rotations about the origin of $\mathbb{R}^3$.

Moreover, this group action is $\mathbb{R}$-linear, so this is our first example of a 'group representation'.

Representation Theory of Groups

Definition

Let $(G, \star)$ be a group and V be a $\Bbbk$-vector space. A ***representation of*** $(G, \star)$ ***on*** V is a group morphism $\rho\colon (G, \star) \to (\mathrm{GL}_{\Bbbk}(V), \circ)$. We say that the representation is **finite-dimensional** when $\dim_{\Bbbk} V < \infty$.

So really, group representations are a special case of group actions.

If $V \cong \Bbbk^n$, then $\mathrm{GL}_{\Bbbk}(V) \cong \mathrm{GL}_{\Bbbk}(\Bbbk^n) \cong \mathrm{GL}_n(\Bbbk)$.

Examples

- $\mathrm{triv}\colon (G, \star) \longrightarrow (\mathrm{GL}_{\mathbb{C}}(\mathbb{C}), \circ) \cong (\mathbb{C}^*, \cdot)$, where $\mathrm{triv}(g) = 1\ \forall g \in G$
- $\chi\colon (\mathbb{Z}/n\mathbb{Z}, +) \longrightarrow (\mathrm{GL}_{\mathbb{C}}(\mathbb{C}), \circ) \cong (\mathbb{C}^*, \cdot)$, where $\chi(\overline{m}) = e^{2\pi i m/n}$
- $\varphi\colon (S_3, \circ) \longrightarrow (\mathrm{GL}_{\mathbb{C}}(\mathbb{C}^2), \circ) \cong (\mathrm{GL}_2(\mathbb{C}), \cdot)$, where

$$\varphi((1\ 2)) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \varphi((1\ 2\ 3)) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

Definition

Given two representations $\rho_1 \colon (G, \star) \to (\mathrm{GL}_{\Bbbk}(V_1), \circ)$ and $\rho_2 \colon (G, \star) \to (\mathrm{GL}_{\Bbbk}(V_2), \circ)$, a **morphism from ρ_1 to ρ_2** is a $\Bbbk$-linear map $T \colon V_1 \to V_2$ such that the following diagram commutes $\forall g \in G$:

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "T"'] & V_1 \arrow[d, "T"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

If T is invertible, we say that T is an **isomorphism from ρ_1 to ρ_2** and write $\rho_1 \cong \rho_2$.

Proposition-Definition

Given two representations $\rho_1\colon (G,\star) \to (\mathrm{GL}_{\Bbbk}(V_1),\circ)$ and $\rho_2\colon (G,\star) \to (\mathrm{GL}_{\Bbbk}(V_2),\circ)$, the map $\rho_1 \oplus \rho_2\colon G \to \mathrm{GL}_{\Bbbk}(V_1 \oplus V_2)$, given by $(\rho_1 \oplus \rho_2)(g)((v_1, v_1)) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$, determines a representation of $(G,\star)$ on $V_1 \oplus V_2$ called the **direct sum representation of ρ_1 and ρ_2**.

Given representations $\rho_1\colon (G,\star) \to (\mathrm{GL}_m(\Bbbk),\cdot)$ and $\rho_2\colon (G,\star) \to (\mathrm{GL}_n(\Bbbk),\cdot)$, their direct sum is the representation $\rho_1 \oplus \rho_2\colon (G,\star) \to (\mathrm{GL}_{m+n}(\Bbbk),\cdot)$, where

$$(\rho_1 \oplus \rho_2)(g) = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$

Example (Permutation Representation)

$\psi\colon (S_n, \circ) \to (\mathrm{GL}_{\mathbb{C}}(\mathbb{C}^n), \circ), \sigma \mapsto \psi_\sigma$, where $\psi_\sigma(e_i) = e_{\sigma(i)}$ and $e_1, \dots, e_n$ are the standard basis vectors of $\mathbb{C}^n$

The subspaces $V_1 = \mathbb{C}(e_1 + \cdots + e_n) = \{\sum_i \lambda_i e_i \mid \lambda_1 = \cdots = \lambda_n\}$ and $V_2 = \{\sum_i \lambda_i e_i \mid \sum_i \lambda_i = 0\}$ are invariant under ψ_σ $\forall \sigma \in S_n$. Moreover, $\mathbb{C}^n = V_1 \oplus V_2$

Therefore, $\psi|_{V_1}\colon (S_n, \circ) \to (\mathrm{GL}_{\mathbb{C}}(V_1), \circ), \sigma \mapsto \psi_\sigma$ and $\psi|_{V_2}\colon (S_n, \circ) \to (\mathrm{GL}_{\mathbb{C}}(V_2), \circ), \sigma \mapsto \psi_\sigma$ are group representations as well

In particular, $\psi \cong \psi|_{V_1} \oplus \psi|_{V_2}$

Definition

Given a representation $\rho\colon (G,\star) \to (\mathrm{GL}_{\Bbbk}(V),\circ)$ and a subspace W of V, we say W is $(G,\star)$-**invariant** if $\rho(g)W \subseteq W \ \forall g \in G$.

In this case, there is an induced representation $\rho|_W : (G,\star) \to (\mathrm{GL}_{\Bbbk}(W),\circ)$ given by $\rho|_W(g) = \rho(g)$.

Definition

A (non-zero) representation $\rho\colon (G,\star) \to (\mathrm{GL}_{\Bbbk}(V),\circ)$ is **irreducible** if the only $(G,\star)$-invariant subspaces of V are $\{0\}$ and V.

Example (Permutation Representation)

$\psi\colon (S_n, \circ) \to (\mathrm{GL}_{\mathbb{C}}(\mathbb{C}^n), \circ), \sigma \mapsto \psi_\sigma$, where $\psi_\sigma(e_i) = e_{\sigma(i)}$ and $e_1, \dots, e_n$ are the standard basis vectors of $\mathbb{C}^n$

The subspaces $V_1 = \mathbb{C}(e_1 + \cdots + e_n) = \{\sum_i \lambda_i e_i \mid \lambda_1 = \cdots = \lambda_n\}$ and $V_2 = \{\sum_i \lambda_i e_i \mid \sum_i \lambda_i = 0\}$ are $(S_n, \circ)$-invariant

Moreover, the representations $\psi|_{V_1}\colon (S_n, \circ) \to (\mathrm{GL}_{\mathbb{C}}(V_1), \circ)$ and $\psi|_{V_2}\colon (S_n, \circ) \to (\mathrm{GL}_{\mathbb{C}}(V_2), \circ)$ are irreducible

So we have a decomposition into irreducibles: $\psi \cong \psi|_{V_1} \oplus \psi|_{V_2}$

Definition

A representation $\rho\colon (G,\star) \to (\mathrm{GL}_{\Bbbk}(V),\circ)$ is **semisimple** if there exists a decomposition $V = V_1 \oplus \cdots \oplus V_n$, where each V_i is $(G,\star)$-invariant and each $\rho|_{V_i}$ is irreducible ($\forall i = 1,\ldots,n$)

Theorem (Maschke)

Every (finite-dimensional) representation of a finite group is semisimple (assuming $\operatorname{char} \Bbbk \nmid |G|$).

So: classifying all possible irreducible (fin-dim) representations of a finite group $(G,\star)$ (up to isomorphism) will classify all possible (fin-dim) representations (up to isomorphism)

Example

Setting $\omega_n = e^{2\pi i/n}$, then $\chi_k \colon (\mathbb{Z}/n\mathbb{Z}, +) \to (\mathbb{C}^*, \cdot)$, $\overline{m} \mapsto \omega_n^{km}$ is a representation for each $k = 1, \ldots, n-1$. The representations $\chi_0, \ldots, \chi_{n-1}$ classify the distinct irreducible representations of $(\mathbb{Z}/n\mathbb{Z}, +)$ up to isomorphism.

Theorem

Let $\{\rho_i : (G, \star) \longrightarrow (\mathrm{GL}_{\Bbbk}(V_i), \circ)\}_{i=1,\ldots,n}$ be all the distinct irreducible representations of a finite group $(G, \star)$ up to isomorphism and let $d_i = \dim_{\Bbbk} V_i$. Then

$$|G| = d_1^2 + \cdots + d_n^2.$$

Moreover, $d_i \mid |G|$ for each $i = 1, \ldots, n$.

Theorem

The number of all distinct irreducible representations of a finite group $(G, \star)$ (up to isomorphism) is equal to the number of conjugacy classes of $(G, \star)$.

Definition

The **tensor product** of two $\Bbbk$-vector spaces V and W is the new $\Bbbk$-vector space $V \otimes W = \mathrm{span}_{\Bbbk}\{v \otimes w \mid v \in V,\ w \in W\}$, where $(-) \otimes (-)$ is $\Bbbk$-bilinear:

$$(\lambda_1 v_1 + \lambda_2 v_2) \otimes w = \lambda_1 (v_1 \otimes w) + \lambda_2 (v_2 \otimes w),$$
$$v \otimes (\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 (v \otimes w_1) + \lambda_2 (v \otimes w_2),$$

where $v, v_1, v_2 \in V, w, w_1, w_2 \in W, \lambda_1, \lambda_2 \in \Bbbk$.

If V has basis $\{a_1, \ldots, a_m\}$ and W has basis $\{b_1, \ldots, b_n\}$, then $V \otimes W$ has basis $\{a_i \otimes b_j \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$.

Proposition-Definition

Given two representations $\rho_1\colon (G,\star) \longrightarrow (\mathrm{GL}_{\Bbbk}(V_1),\circ)$ and $\rho_2\colon (G,\star) \longrightarrow (\mathrm{GL}_{\Bbbk}(V_2),\circ)$, the map $\rho_1 \otimes \rho_2\colon G \longrightarrow \mathrm{GL}_{\Bbbk}(V_1 \otimes V_2)$, given by $(\rho_1 \otimes \rho_2)(g)((v_1 \otimes v_2)) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$, determines a representation of $(G,\star)$ on $V_1 \otimes V_2$ called the **tensor product representation of ρ_1 and ρ_2.**

A famous theorem in group theory proven using representation theory (no alternative proof was found until the 1970's):

Burnside's Theorem

Let $(G, \star)$ be a group of order $p^a q^b$, where p and q are prime. Then $(G, \star)$ is solvable.

The dream:

- Classify all irreducible representations
- There has been success with more well-understood algebraic objects when restricting to finite-dimensional representations
- What about for infinite dimensional representations? Not really

Representation Theory of Associative Algebras

So, what next?

Definition

A (unital, associative) $\Bbbk$-algebra $A = (A, +, \cdot)$ is a $\Bbbk$-vector space $(A, +)$ such that:

- $\exists e \in A$ such that $e \cdot a = a = a \cdot e \ \forall a \in A$ (usually, we denote $e = 1_A = 1$)
- $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b) \ \forall a, b \in A, \forall \lambda \in \Bbbk$
- $(a \cdot b) \cdot c = a \cdot (b \cdot c) \ \forall a, b, c \in A$
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c) \ \forall a, b, c \in A$
- $(b + c) \cdot a = (b \cdot a) + (c \cdot a) \ \forall a, b, c \in A$

Examples

- $(\mathbb{k}, +, \cdot)$ is a $\mathbb{k}$-algebra
- The polynomial ring $(\mathbb{k}[x_1, \ldots, x_n], +, \cdot)$ is a $\mathbb{k}$-algebra
- For a group $(G, \star)$, the group algebra $(\mathbb{k}[G], +, \star)$ is a $\mathbb{k}$-algebra
- For a complex Lie algebra $\mathfrak{g}$, the universal enveloping algebra $(\mathfrak{U}(\mathfrak{g}), +, \cdot)$ is a $\mathbb{C}$-algebra
- For a $\mathbb{k}$-vector space, $(\mathrm{End}_{\mathbb{k}}(V) = \{\varphi\colon V \to V \mid \varphi \text{ is } \mathbb{k}\text{-linear}\}, +, \circ)$ is a $\mathbb{k}$-algebra
- $(M_n(\mathbb{k}), +, \circ)$ is a $\mathbb{k}$-algebra

Definition

If $(A, +, \cdot)$ and $(B, +, \cdot)$ are $\Bbbk$-algebras, then an **algebra morphism** $\rho\colon (A, +, \cdot) \longrightarrow (B, +, \cdot)$ is a $\Bbbk$-linear map $\rho\colon A \longrightarrow B$ such that

- $\rho(1_A) = 1_B$
- $\rho(a_1 a_2) = \rho(a_1)\rho(a_2)\ \forall a_1, a_2 \in A$

Definition

Let $(A, +, \cdot)$ be a $\Bbbk$-algebra and V be a $\Bbbk$-vector space. A **representation of** $(A, +, \cdot)$ **on** V is an algebra morphism $\rho\colon (A, +, \cdot) \longrightarrow (\mathrm{End}_{\Bbbk}(V), +, \circ)$.

An algebra rep $\varphi\colon (A,+,\cdot) \to (\mathrm{End}_{\Bbbk}(V),+,\circ) \;=\; V$ is a (left) A-module

A group rep $\rho\colon (G,\star) \to (\mathrm{GL}_{\Bbbk}(V),\circ) \;=\; V$ is a (left) $\Bbbk[G]$-module

A Lie algebra rep $\psi\colon (\mathfrak{g},+,[\cdot,\cdot]) \to (\mathfrak{gl}_{\Bbbk}(V),+,[\cdot,\cdot]) \;=\; V$ is a (left) $\mathfrak{U}(\mathfrak{g})$-module

So representation theory is a study of module theory

Similar machinery from group representations are available for algebra representations, such as direct products

However a tensor product of algebra representations $\rho_1 \colon (A, +, \cdot) \to (\mathrm{End}_{\Bbbk}(V_1), +, \circ)$, $\rho_2 \colon (A, +, \cdot) \to (\mathrm{End}_{\Bbbk}(V_2), +, \circ)$ will not be a representation of A, but rather of $A \otimes A$

Algebras for which the tensor product of its representations is again a representation of itself are called **Hopf algebras**

Quantum groups are important examples of Hopf algebras

End